

# On Soliton Dynamics in Nonlinear Schrödinger Equations\*

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## Abstract

In this paper we announce the result of asymptotic dynamics of solitons of nonlinear Schrödinger equations with external potentials. To each local minima of the potential there is a soliton centered around it. Under some conditions on the nonlinearity, the potential and the datum, we prove that the solution can be decomposed into two parts: the soliton and the term dissipating to infinity.

## 1 Introduction

**Problem.** In this paper we study dynamics of solitons in the generalized nonlinear Schrödinger equation (NLS) in dimension  $d \neq 2$  with an external potential  $V_h : \mathbb{R}^n \rightarrow \mathbb{R}$ ,

$$i \frac{\partial \psi}{\partial t} = -\Delta \psi + V_h \psi - f(|\psi|^2) \psi. \quad (1)$$

Here  $h > 0$  is a small parameter giving the length scale of the external potential in relation to the length scale of the  $V_h = 0$  solitons (see below),  $\Delta$  is the Laplace operator and  $f(s)$  is a nonlinearity to be specified later. We normalize  $f(0) = 0$ . Such equations arise in the theory of Bose-Einstein condensation<sup>1</sup>, nonlinear optics, theory of water waves<sup>2</sup> and in other areas.

To fix ideas we assume the potentials to be of the form  $V_h(x) := V(hx)$  with  $V$  smooth and decaying at  $\infty$ . Thus for  $h = 0$ , Equation (1) becomes the standard generalized nonlinear Schrödinger equation (gNLS)

$$i \frac{\partial \psi}{\partial t} = -\Delta \psi + \mu \psi - f(|\psi|^2) \psi, \quad (2)$$

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<sup>1</sup>In this case Equation (1) is called the Gross-Pitaevskii equation.

<sup>2</sup>In these two areas  $V_h$  arises if one takes into account impurities and/or variations in geometry of the medium and is, in general, time-dependent.

where  $\mu = V(0)$ . For a certain class of nonlinearities,  $f(|\psi|^2)$  (see Section 3), there is an interval  $\mathcal{I}_0 \subset \mathbb{R}$  such that for any  $\lambda \in \mathcal{I}_0$  Equation (2) has solutions of the form  $e^{i(\lambda-\mu)t} \phi_0^\lambda(x)$  where  $\phi_0^\lambda \in \mathcal{H}_2(\mathbb{R}^n)$  and  $\phi_0^\lambda > 0$ . Such solutions (in general without the restriction  $\phi_0^\lambda > 0$ ) are called the *solitary waves* or *solitons* or, to emphasize the property  $\phi_0^\lambda > 0$ , the *ground states*. For brevity we will use the term *soliton* applying it also to the function  $\phi_0^\lambda$  without the phase factor  $e^{i(\lambda-\mu)t}$ .

Equation (2) is translationally and gauge invariant. Hence if  $e^{i(\lambda-\mu)t} \phi_0^\lambda(x)$  is a solution for Equation (2), then so is

$$e^{i(\lambda-\mu)t} e^{i\alpha} \phi_0^\lambda(x+a), \text{ for any } a \in \mathbb{R}^n, \text{ and } \alpha \in [0, 2\pi).$$

This situation changes dramatically when the potential  $V_h$  is turned on. In general, as was shown in [FW, Oh1] out of the  $(n+2)$ -parameter family  $e^{i(\lambda-\mu)t} e^{i\alpha} \phi_0^\lambda(x+a)$  only a discrete set of two-parameter families of solutions to Equation (1) bifurcate:  $e^{i\lambda t} e^{i\alpha} \phi^\lambda(x)$ ,  $\alpha \in [0, 2\pi)$  and  $\lambda \in \mathcal{I}$  for some  $\mathcal{I} \subseteq \mathcal{I}_0$ , with  $\phi^\lambda \equiv \phi_h^\lambda \in \mathcal{H}_2(\mathbb{R}^n)$  and  $\phi^\lambda > 0$ . Each such family centers near a different critical point of the potential  $V_h(x)$ . It was shown in [Oh2] that the solutions corresponding to minima of  $V_h(x)$  are orbitally (Lyapunov) stable and to maxima, orbitally unstable. We call the solitary wave solutions described above which correspond to the minima of  $V_h(x)$  *trapped solitons* or just *solitons* of Equation (1) omitting the last qualifier if it is clear which equation we are dealing with.

**Results.** In this note we describe results of [GS1, GS2] that the trapped solitons of Equation (1) are asymptotically stable. The latter property means that if an initial condition of (1) is sufficiently close to a trapped soliton then the solution converges (relaxes),

$$\psi(x, t) - e^{i\gamma(t)} \phi^{\lambda_0}(x) \rightarrow 0,$$

in some weighted  $\mathcal{L}^2$  space to, in general, another trapped soliton of the same two-parameter family. We also find effective equations for the soliton center and other parameters. In this paper we prove this result under the additional assumption that if  $d > 2$  then the potential is spherically symmetric and that the initial condition symmetric with respect to permutations of the coordinates. In this case the soliton relaxes to the ground state along the radial direction. This limits the number of technical difficulties we have to deal with. We expect that our techniques extend to the general case when the soliton spirals toward its equilibrium.

In fact, [GS1, GS2] prove a result more general than asymptotic stability of trapped solitons. Namely, we show that if an initial condition is close (in the weighted norm  $\|u\|_{\nu,1} := \|(1+|x|^2)^{\frac{\nu}{2}} u\|_{\mathcal{H}^1}$  for sufficiently large  $\nu$ ) to the soliton  $e^{i\gamma_0} \phi^{\lambda_0}$ , with  $\gamma_0 \in \mathbb{R}$  and  $\lambda_0 \in \mathcal{I}$  ( $\mathcal{I}$  as above), then the solution,  $\psi(t)$ , of Equation (1) can be written as

$$\psi(x, t) = e^{i\gamma(t)} (e^{ip(t) \cdot x} \phi^{\lambda(t)}(x - a(t)) + R(x, t)), \quad (3)$$

where  $\|R(t)\|_{-\nu,1} \rightarrow 0$ ,  $\lambda(t) \rightarrow \lambda_\infty$  for some  $\lambda_\infty$  as  $t \rightarrow \infty$  and the soliton center  $a(t)$  and momentum  $p(t)$  evolve according to an effective equations of motion close to Newton's equation in the potential  $h^2V(a)$ .

We observe that (1) is a Hamiltonian system with conserved energy (see Section 2) and, though orbital (Lyapunov) stability is expected, the asymptotic stability is a subtle matter. To have asymptotic stability the system should be able to dispose of excess of its energy, in our case, by radiating it to infinity. The infinite dimensionality of a Hamiltonian system in question plays a crucial role here. This phenomenon as well as a general class of classical and quantum relaxation problems was pointed out by J. Fröhlich and T. Spencer [Private Communication].

We also mention that because of slow time-decay of the linearized propagator, the low dimensions  $d = 1, 2$  are harder to handle than the higher dimensions,  $d > 2$ .

**Previous results.** We refer to [GS1] for a detailed review of the related literature. Here we only mention results of [Cu, BP1, BP2, BuSu, SW1, SW2, SW3, TY1, TY2, TY3] which deal with a similar problem. Like our work, [SW1, SW2, SW3, TY1, TY2, TY3] study the ground state of the NLS with a potential. However, these papers deal with the near-linear regime in which the nonlinear ground state is a bifurcation of the ground state for the corresponding Schrödinger operator  $-\Delta + V(x)$ . The present paper covers highly nonlinear regime in which the ground state is produced by the nonlinearity (our analysis simplifies considerably in the near-linear case). Now, papers [Cu, BP1, BP2, BuSu] consider the NLS without a potential so the corresponding solitons, which were described above, are affected only by a perturbation of the initial conditions which disperses with time leaving them free. While in our case they, in addition, are under the influence of the potential and they relax to an equilibrium state near a local minimum of the potential.

**Open problems.** We formulate some open problems:

- (1) Extend the results of this present paper to more general initial conditions and to more general, probably time-dependent, potentials.
- (2) Link the results of this paper with the results of [FGJS] on the long time dynamics of solitons.

A natural place to start here is spherically symmetric potentials but general initial conditions. Note that for certain time-dependent potentials the solitons will never settle in the ground state.

**Notation.** As customary we often denote derivatives by subindices as in  $\phi_\lambda^\lambda = \frac{\partial}{\partial \lambda} \phi^\lambda$  for  $\phi^\lambda = \phi^\lambda(x)$ . However, the subindex  $h$  signifies always the dependence on the parameter  $h$  and not the derivatives in  $h$ . The Sobolev and  $L^2$  spaces are denoted by  $\mathcal{H}^k$  and  $\mathcal{L}^2$  respectively.

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## 2 Hamiltonian Structure and GWP

Equation (1) is a Hamiltonian system on Sobolev space  $\mathcal{H}^1(\mathbb{R}, \mathbb{C})$  viewed as a real space  $\mathcal{H}^1(\mathbb{R}, \mathbb{R}) \oplus \mathcal{H}^1(\mathbb{R}, \mathbb{R})$  with the inner product  $(\psi, \phi) = \operatorname{Re} \int_{\mathbb{R}} \bar{\psi} \phi$  and with the symplectic form  $\omega(\psi, \phi) = \operatorname{Im} \int_{\mathbb{R}} \bar{\psi} \phi$ . The Hamiltonian functional is:

$$H(\psi) := \int \left[ \frac{1}{2} (|\psi_x|^2 + V_h |\psi|^2) - F(|\psi|^2) \right],$$

where  $F(u) := \frac{1}{2} \int_0^u f(\xi) d\xi$ .

Equation (1) has the time-translational and gauge symmetries which imply the following conservation laws: for any  $t \geq 0$ , we have

(CE) conservation of energy:  $H(\psi(t)) = H(\psi(0))$ ;

(CP) conservation of the number of particles:  $N(\psi(t)) = N(\psi(0))$ , where  $N(\psi) := \int |\psi|^2$ .

To address the global well-posedness of (1) we need the following condition on the nonlinearity  $f$ . Below,  $s_+ = s$  if  $s > 0$  and  $= 0$  if  $s \leq 0$ .

(fA) The nonlinearity  $f$  satisfies the estimate  $|f'(\xi)| \leq c(1 + |\xi|^{\alpha-1})$  for some  $\alpha \in [0, \frac{2}{(d-2)_+})$  and  $|f(\xi)| \leq c(1 + |\xi|^\beta)$  for some  $\beta \in [0, \frac{2}{d})$ .

The following result can be found in [Caz].

**Theorem** *Assume that the nonlinearity  $f$  satisfies the condition (fA), and that the potential  $V$  is bounded. Then Equation (1) is globally well posed in  $\mathcal{H}^1$ , i.e. the Cauchy problem for Equation (1) with a datum  $\psi(0) \in \mathcal{H}^1$  has a unique solution  $\psi(t)$  in the space  $\mathcal{H}^1$  and this solution depends continuously on  $\psi(0)$ . Moreover  $\psi(t)$  satisfies the conservation laws (CE) and (CP).*

## 3 Existence and Orbital Stability of Solitons

In this section we review the question of existence of the solitons (ground states) for Equation (1). Assume the nonlinearity  $f : \mathbb{R} \rightarrow \mathbb{R}$  is smooth and satisfies

(fB) There exists an interval  $\mathcal{I}_0 \subset \mathbb{R}^+$  s.t. for any  $\lambda \in \mathcal{I}_0$ ,  $-\infty \leq \overline{\lim}_{s \rightarrow +\infty} \frac{f(s)}{s^{\frac{2}{d-2}}} \leq 0$  and  $\frac{1}{\xi} \int_0^\xi f(s) ds > \lambda$  for some constant  $\xi$ , for  $d > 2$ ; and

$$U(\phi, \lambda) := -\lambda \phi^2 + \int_0^{\phi^2} f(\xi) d\xi$$

has a smallest positive root  $\phi_0(\lambda)$  such that  $U_\phi(\phi_0(\lambda), \lambda) \neq 0$ , for  $d = 1$ .

It is shown in [BL, Str] that under Condition (fB) there exists a spherical symmetric positive solution  $\phi^\lambda$  to the equation

$$-\Delta \phi^\lambda + \lambda \phi^\lambda - f((\phi^\lambda)^2) \phi^\lambda = 0. \quad (4)$$

**Remark 1.** *Existence of soliton functions  $\phi^\lambda$  for  $d = 2$  is proved in [Str] under different conditions on  $f$ .*

When the potential  $V$  is present, then some of the solitons above bifurcate into solitons for Equation (1). Namely, let, in addition,  $f$  satisfy the condition  $|f'(\xi)| \leq c(1 + |\xi|^p)$ , for some  $p < \infty$ , and  $V$  satisfy the condition

(VA)  $V$  is smooth and 0 is a non-degenerate local minimum of  $V$ .

Then, similarly as in [FW, Oh1] one can show that if  $h$  is sufficiently small, then for any  $\lambda \in \mathcal{I}_{0V}$ , where

$$\mathcal{I}_{0V} := \{\lambda | \lambda > -\inf_{x \in \mathbb{R}} \{V(x)\}\} \cap \{\lambda | \lambda + V(0) \in \mathcal{I}_0\},$$

there exists a unique soliton  $\phi^\lambda \equiv \phi_h^\lambda$  (i. e.  $\phi^\lambda \in \mathcal{H}_2(\mathbb{R})$  and  $\phi^\lambda > 0$ ) satisfying the equation

$$-\Delta \phi^\lambda + (\lambda + V_h) \phi^\lambda - f((\phi^\lambda)^2) \phi^\lambda = 0$$

and the estimate  $\phi^\lambda = \phi_0^{\lambda+V(0)} + O(h^{3/2})$  where  $\phi_0^\lambda$  is the soliton of Equation (4).

Let  $\delta(\lambda) := \|\phi^\lambda\|_2^2$ . It is shown in [GSS1] that the soliton  $\phi^\lambda$  is a minimizer of the energy functional  $H(\psi)$  for a fixed number of particles  $N(\psi) = \text{constant}$  if and only if  $\delta'(\lambda) > 0$ . Moreover, it is shown in [We2, GSS1] that under the latter condition the solitary wave  $\phi^\lambda e^{i\lambda t}$  is orbitally stable. Under more restrictive conditions (see [GSS1]) on  $f$  one can show that the open set

$$\mathcal{I} := \{\lambda \in \mathcal{I}_{0V} : \delta'(\lambda) > 0\} \quad (5)$$

is non-empty. Instead of formulating these conditions we assume in what follows that the open set  $\mathcal{I}$  is non-empty and  $\lambda \in \mathcal{I}$ .

Using the equation for  $\phi^\lambda$  one can show that if the potential  $V$  is radially symmetric then there exist constants  $c, \delta > 0$  such that  $|\phi^\lambda(x)| \leq ce^{-\delta|x|}$  and  $|\frac{d}{d\lambda}\phi^\lambda| \leq ce^{-\delta|x|}$ , and similarly for the derivatives of  $\phi^\lambda$  and  $\frac{d}{d\lambda}\phi^\lambda$ .

## 4 Linearized Equation and Resonances

We rewrite Equation (1) as  $\frac{d\psi}{dt} = G(\psi)$  where the nonlinear map  $G(\psi)$  is defined by  $G(\psi) = -i(-\Delta + \lambda + V_h)\psi + if(|\psi|^2)\psi$ . Then the linearization of Equation (1) can be written as  $\frac{\partial \chi}{\partial t} = \partial G(\phi^\lambda)\chi$  where  $\partial G(\phi^\lambda)$  is the Fréchet derivative of  $G(\psi)$  at  $\phi$ . It is computed to be

$$\partial G(\phi^\lambda)\chi = -i(-\Delta + \lambda + V_h)\chi + if'((\phi^\lambda)^2)\chi + 2if'((\phi^\lambda)^2)(\phi^\lambda)^2 \text{Re}\chi. \quad (6)$$

This is a real linear but not complex linear operator. To convert it to a linear operator we pass from complex functions to real vector-functions  $\chi \longleftrightarrow \vec{\chi} =$

$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$ , where  $\chi_1 = \text{Re}\chi$  and  $\chi_2 = \text{Im}\chi$ . Then  $\partial G(\phi^\lambda)\chi \longleftrightarrow L(\lambda)\vec{\chi}$  where the operator  $L(\lambda)$  is given by

$$L(\lambda) := \begin{pmatrix} 0 & L_-(\lambda) \\ -L_+(\lambda) & 0 \end{pmatrix}, \quad (7)$$

with  $L_-(\lambda) := -\Delta + V_h + \lambda - f((\phi^\lambda)^2)$ , and  $L_+(\lambda) := -\Delta + V_h + \lambda - f((\phi^\lambda)^2) - 2f'((\phi^\lambda)^2)(\phi^\lambda)^2$ . The operator  $L(\lambda)$  is extended to the complex space  $\mathcal{H}^2(\mathbb{R}, \mathbb{C}) \oplus \mathcal{H}^2(\mathbb{R}, \mathbb{C})$ . If the potential  $V_h$  in Equation (1) decays at  $\infty$ , then by a general result

$$\sigma_{ess}(L(\lambda)) = (-i\infty, -i\lambda] \cap [i\lambda, i\infty).$$

The eigenfunctions of  $L(\lambda)$  are described in the following theorem (cf [GS1], [GS2]).

**Theorem 4.1.** *Let  $V$  satisfy Condition (VA) and  $|h|$  be sufficiently small. Then the operator  $L(\lambda)$  has at least  $2d + 2$  eigenvectors and associated eigenvalues with eigenvalues near zero: two-dimensional space with the eigenvalue 0 and a 2d-dimensional space with non-zero imaginary eigenvalues  $\pm i\epsilon_j(\lambda)$ ,*

$$\epsilon_j(\lambda) := h\sqrt{2e_j} + o(h),$$

where  $e_j$  are eigenvalues of the Hessian matrix of  $V$  at value  $x = 0$ ,  $V''(0)$ . The corresponding eigenfunctions  $\begin{pmatrix} \xi_j \\ \pm i\eta_j \end{pmatrix}$  are related by complex conjugation and satisfy

$$\xi_j = \sqrt{2}\partial_{x_k}\phi_0^\lambda + o(h) \text{ and } \eta_j = -h\sqrt{e_j}x_j\phi_0^\lambda + o(h),$$

and  $\xi_i$  and  $\eta_j$  are real.

**Remark 2.** *The zero eigenvector  $\begin{pmatrix} 0 \\ \phi^\lambda \end{pmatrix}$  and the associated zero eigenvector  $\begin{pmatrix} \partial_\lambda \phi^\lambda \\ 0 \end{pmatrix}$  are related to the gauge symmetry  $\psi(x, t) \rightarrow e^{i\alpha}\psi(x, t)$  of the original equation and the 2d eigenvectors  $\begin{pmatrix} \xi_j \\ \pm i\eta_j \end{pmatrix}$  with  $O(h)$  eigenvalues originate from the zero eigenvectors  $\begin{pmatrix} \partial_{x_k}\phi_0^\lambda \\ 0 \end{pmatrix}$ ,  $k = 1, 2, \dots, d$ , and the associated zero eigenvectors  $\begin{pmatrix} 0 \\ x_k\phi_0^\lambda \end{pmatrix}$ ,  $k = 1, 2, \dots, d$ , of the  $V = 0$  equation due to the translational symmetry and to the boost transformation  $\psi(x, t) \rightarrow e^{ib \cdot x}\psi(x, t)$  (coming from the Galilean symmetry), respectively.*

For  $d \geq 2$  we will be interested in permutationally symmetric functions,  $g \in \mathcal{L}^2(\mathbb{R}^d)$ , characterized as

$$g(x) = g(\sigma x) \text{ for any } \sigma \in S_d$$

with  $S_d$  being the group of permutation of  $d$  indices and  $\sigma(x_1, x_2, \dots, x_d) := (x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(d)})$ .

**Remark 3.** For any function of the form  $e^{ip \cdot x} \phi(|x - a|)$  with  $a \parallel p$ , there exists a rotation  $\tau$  such that the function  $e^{ip \cdot \tau x} \phi(|\tau x - a|) = e^{i\tau^{-1}p \cdot x} \phi(|x - \tau^{-1}a|)$  is permutationally symmetric. Such families describe wave packets with the momenta directed toward or away from the origin.

If for  $d \geq 2$  the potential  $V(x)$  is spherically symmetric, then  $V''(0) = \frac{1}{d} \Delta V(0) \cdot Id_{n \times n}$ , and therefore the eigenvalues  $e_j$  of  $V''(0)$  are all equal to  $\frac{1}{d} \Delta V(0)$ . Thus we have

**Corollary 4.2.** Let  $d \geq 2$  and  $V$  satisfy Condition (VA) and let  $V$  be spherically symmetric. Then  $L(\lambda)$  restricted to permutational symmetric functions has 4 eigenvectors or associated eigenvectors near zero: two-dimensional space with eigenvalue 0; and two-dimensional space with the non-zero imaginary eigenvalues  $\pm i\epsilon(\lambda)$ , where

$$\epsilon(\lambda) = h \sqrt{\frac{2\Delta V(0)}{d}} + o(h),$$

and with the eigenfunctions  $\begin{pmatrix} \xi(\lambda) \\ \pm i\eta(\lambda) \end{pmatrix}$ , where  $\xi$  and  $\eta$  are real, and permutation symmetric functions satisfying

$$\xi(\lambda) = \sqrt{2} \sum_{n=1}^d \frac{d}{dx_n} \phi_0^\lambda + O(h) \text{ and } \eta(\lambda) = -h \sqrt{\frac{1}{d} \Delta V(0)} \sum_{n=1}^d x_n \phi_0^\lambda + O(h^{3/2}).$$

The eigenvectors  $\begin{pmatrix} \xi(\lambda) \\ \pm i\eta(\lambda) \end{pmatrix}$  are symmetric combinations of the eigenvectors described in Theorem 2.

Besides eigenvalues, the operator  $L(\lambda)$  may have resonances at the tips,  $\pm i\lambda$ , of its essential spectrum (those tips are called thresholds). Recall the notation  $\alpha_+ := \alpha$  if  $\alpha > 0$  and  $= 0$  if  $\alpha \leq 0$ .

**Definition 4.3.** Let  $d \neq 2$ . A function  $h$  is called a resonance function of  $L(\lambda)$  at  $\mu = \pm i\lambda$  if  $h \notin \mathcal{L}^2$ ,  $|h(x)| \leq c \langle x \rangle^{-(d-2)_+}$  and  $h$  is  $C^2$  and solves the equation

$$(L(\lambda) - \mu)h = 0.$$

Note that this definition implies that for  $d > 2$  the resonance function  $h$  solves the equation  $(1 + K(\lambda))h = 0$  where  $K(\lambda)$  is a family of compact operators given by  $K(\lambda) := (L_0(\lambda) - \mu + 0)^{-1} V_{big}(\lambda)$ . Here  $L_0(\lambda) := \begin{pmatrix} 0 & -\Delta + \lambda \\ \Delta - \lambda & 0 \end{pmatrix}$  and

$$V_{big}(\lambda) := \begin{pmatrix} 0 & V_h - f((\phi^\lambda)^2) \\ -V_h + f((\phi^\lambda)^2) + 2f'((\phi^\lambda)^2)(\phi^\lambda)^2 & 0 \end{pmatrix}. \quad (8)$$

In this paper we make the following assumptions on the point spectrum and resonances of the operator  $L(\lambda)$ :

(SA)  $L(\lambda)$  has only 4 standard and associated eigenvectors in the permutation symmetric subspace.

(SB)  $L(\lambda)$  has no resonances at  $\pm i\lambda$ .

The discussion and results concerning these conditions, given in [GS1], suggested strongly that Condition (SA) is satisfied for a large class of nonlinearities and potentials and Condition (SB) is satisfied generically. Elsewhere we show this using earlier results of [CP, CPV]. We also assume the following condition

(FGR) Let  $N$  be the smallest positive integer such that  $\epsilon(\lambda)(N+1) > \lambda$ ,  $\forall \lambda \in I$ . Then  $ReY_N < 0$  where  $Y_n$ ,  $n = 1, 2, \dots$ , are the functions of  $V$  and  $\lambda$ , defined in Equations (17) below (see also (12)).

We expect that Condition (FGR) holds generically. Theorem 5.2 below shows that  $ReY_n = 0$  if  $n < N$ .

We expect the following is true:

(a) if for some  $N_1(\geq N)$ ,  $ReY_n = 0$  for  $n < N_1$ , then  $ReY_{N_1} \leq 0$  and (b) for generic potentials/nonlinearities there exists an  $N_1(\geq N)$  such that  $ReY_{N_1} \neq 0$ . Thus Condition (FGR) could have been generalized by assuming that  $ReY_{N_1} < 0$  for some  $N_1 \geq N$  such that  $ReY_n = 0$  for  $n < N_1$ . We took  $N = N_1$  in order not to complicate the exposition.

The following form of  $ReY_N$

$$ReY_N = Im\langle \sigma_1(L(\lambda) - (N+1)i\epsilon(\lambda) - 0)^{-1}F, F \rangle \leq 0 \quad (9)$$

for some function  $F$  depending on  $\lambda$  and  $V$  and  $\sigma_1 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ , is proved in [BuSu, TY1, TY2, TY3, SW4] for  $N = 1$ , and in [G] for  $N = 2, 3$ . We conjecture that this formula holds for any  $N$ .

Condition (FGR) is related to the Fermi Golden Rule condition which appears whenever time-(quasi)periodic, spatially localized solutions become coupled to radiation. In the standard case it says that this coupling is effective in the second order ( $N = 1$ ) of the perturbation theory and therefore it leads to instability of such solutions. In our case these time-periodic solutions are stationary solutions

$$c_1 \begin{pmatrix} \xi \\ i\eta \end{pmatrix} e^{i\epsilon(\lambda)t} + c_2 \begin{pmatrix} \xi \\ -i\eta \end{pmatrix} e^{-i\epsilon(\lambda)t}$$

of the linearized equation  $\frac{\partial \vec{\chi}}{\partial t} = L(\lambda)\vec{\chi}$  and the coupling is realized through the nonlinearity. Since the radiation in our case is "massive" – the essential spectrum of  $L(\lambda)$  has the gap  $(-i\lambda, i\lambda)$ ,  $\lambda > 0$ , – the coupling occurs only in the  $N$ -th order of perturbation theory where  $N$  is given in Condition (FGR).

The rigorous form of the Fermi Golden Rule for the linear Schrödinger equations was introduced in [BS]. For nonlinear waves and Schrödinger equations the Fermi Golden Rule and the corresponding condition were introduced in [S] and, in the present context, in [SW4, BuSu, BP2, TY1, TY2, TY3].



## 5 Main Results

In this section we state the main theorem of this paper. For technical reason we impose the following conditions on  $f$  and  $V$

(fC) the nonlinearity  $f$  is a smooth function satisfying  $f''(0) = f'''(0) = 0$  if  $d \geq 3$ ; and  $f^{(k)}(0) = 0$  for  $k = 2, 3 \dots 3N + 1$  if  $d = 1$ , where  $f^{(k)}$  is the  $k$ -th derivative of  $f$ , and  $N$  is the same as in Condition (FGR),

(VB)  $V$  decays exponentially fast at  $\infty$ .

**Theorem 5.1.** *Let Conditions (fA)-(fC), (VA), (VB), (SA), (SB) and (FGR) be satisfied and let, for  $d \geq 3$ , the potential  $V$  be spherically symmetric. Let an initial condition  $\psi_0$  be permutation symmetric if  $d \geq 3$  and  $\lambda \in \mathcal{I}$ . There exists  $c, \epsilon_0 > 0$  such that, if*

$$\inf_{\gamma \in \mathbb{R}} \{ \|\psi_0 - e^{i\gamma}(\phi^\lambda + z_1^{(0)}\xi + iz_2^0\eta)\|_{\mathcal{H}^k} + \|(1+x^2)^\nu [\psi_0 - e^{i\gamma}(\phi^\lambda + z_1^{(0)}\xi + iz_2^0\eta)]\|_2 \} \leq c|(z_1^0, z_2^0)|^2 \quad (10)$$

with  $|(z_1^0, z_2^0)| \leq \epsilon_0$  and  $z_n^0$   $n = 1, 2$  being real, some large constant  $\nu > 0$  and with  $k = [\frac{d}{2}] + 2$  if  $d \geq 3$ , and  $k = 1$  if  $d = 1$ , then there exist differentiable functions  $\gamma, z_1, z_2 : \mathbb{R}^+ \rightarrow \mathbb{R}$ ,  $\lambda : \mathbb{R}^+ \rightarrow \mathcal{I}$  and  $R : \mathbb{R}^+ \rightarrow \mathcal{H}^k$  such that the solution,  $\psi(t)$ , to Equation (1) is of the form

$$\psi(t) = e^{i \int_0^t \lambda(s) ds + i\gamma(t)} [\phi^{\lambda(t)} + z_1(t)\xi + iz_2(t)\eta + R(t)] \quad (11)$$

with the following estimates:

(A)  $\|(1+x^2)^{-\nu} R(t)\|_2 \leq c(1+|t|)^{-\frac{1}{2N}}$  where  $\nu$  and  $N$  are the same as that in (10) and (FGR) respectively,

$$(B) \sum_{j=1}^2 |z_j(t)| \leq c(1+t)^{-\frac{1}{2N}}.$$

**Remark 4.** Recall from Remark 3 that the class of permutationally symmetric data includes wave packets with initial momenta directed toward or in the opposite direction of the origin.

**Theorem 5.2.** Under the conditions of Theorem 3 we have

(A) there exists a constant  $\lambda_\infty \in \mathcal{I}$  such that  $\lim_{t \rightarrow \infty} \lambda(t) = \lambda_\infty$ .

(B) Let  $z := z_1 - iz_2$ . Then there exists a change of variables  $\beta = z + O(|z|^2)$  such that

$$\dot{\beta} = i\epsilon(\lambda)\beta + \sum_{n=1}^N Y_n(\lambda)\beta^{n+1}\bar{\beta}^n + O(|\beta|^{2N+2}) \quad (12)$$

with  $Y_n$  being purely imaginary if  $n < N$  and, by Condition (FGR)  $\text{Re}Y_N < 0$ . Moreover, for  $N = 1, 2, 3$ ,  $\text{Re}Y_N$  is given by Equation (9).

**Remark 5.** Using that  $\epsilon(\lambda) = h\sqrt{\frac{2\Delta V(0)}{d}} + o(h)$  one can rewrite Equations ( 11) and ( 12) in the form ( 3) with  $a(t)$  and  $p(t)$  satisfying the equations  $\frac{1}{2}\dot{a} = p$  and  $\dot{p} = -h^2\nabla V(a)$  modulo  $O(|a|^2 + |p|^2)$ .

## 6 Idea of the Proof: $z$ -Expansions

We follow [G, GS2]. We decompose of the solution  $\psi(t)$  to Equation ( 1) into a solitonic component and a symplectically orthogonal fluctuation as (cf [FGJS])

$$\psi(t) = e^{i \int_0^t \lambda(s) ds + i\gamma(t)} (\phi^\lambda + z_1(t)\xi + iz_2(t)\eta + R(t)), \quad (13)$$

where  $\lambda, \gamma, z_1, z_2$  are real, differentiable functions of  $t$  and the function  $R(t)$ , called the *fluctuation*, satisfies the orthogonality conditions

$$\text{Im}\langle R, i\phi^\lambda \rangle = \text{Im}\langle R, \frac{d}{d\lambda}\phi^\lambda \rangle = \text{Im}\langle R, i\eta \rangle = \text{Im}\langle R, \xi \rangle = 0. \quad (14)$$

We plug Equation ( 13) into Equation ( 1) to obtain equations for the parameters  $z_1(t), z_2(t), \lambda(t)$  and  $\gamma(t)$  and the fluctuation  $R(t)$ . As was already discussed above the linearized operator,  $G(\phi^\lambda)$  (see ( 6)), in the equation for  $R(t)$  is only real-linear and therefore we pass from the unknown  $R$  to the unknown  $\vec{R} := \begin{pmatrix} \text{Re} R \\ \text{Im} R \end{pmatrix} \leftrightarrow R$ . Under this correspondence the multiplication by  $i^{-1}$  goes over to the symplectic matrix  $J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : J\vec{R} \leftrightarrow i^{-1}R$ . Unlike with the equation for  $R$ , in the equations for  $z_1$  and  $z_2$  it is more convenient to go from the real, symplectic structure given by  $J$  to the complex structure  $i^{-1}$  by passing from  $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$  to  $z := z_1 - iz_2$ .

A key point is to look for the fluctuation  $\vec{R}(t)$  in the form ( [G, GS2])

$$\vec{R} = \sum_{2 \leq m+n \leq N} \tilde{R}_{m,n}(\lambda) z^m \bar{z}^n + \tilde{R}_N \quad (15)$$

with the remainder  $\tilde{R}_N$  of the order  $O(|z|^{N+1})$ . This leads to the following equations on the coefficients  $\tilde{R}_{mn}$ :

$$[L(\lambda) - i\epsilon(\lambda)(m-n)]\tilde{R}_{mn}(\lambda) = -P_c \tilde{f}_{m,n}(\lambda),$$

where the functions  $\tilde{f}_{m,n}(\lambda)$  depend on  $\tilde{R}_{m',n'}(\lambda)$  with  $m' + n' < m + n$ . Recall that if  $|m - n| \leq N$ , then  $i\epsilon(\lambda)(m - n) \notin \sigma(L(\lambda))$  and therefore the operators

$$L(\lambda) - i\epsilon(\lambda)(m - n) : P_c \mathcal{L}^2 \rightarrow P_c \mathcal{L}^2 \quad (16)$$

are invertible. Hence the above equations have unique solutions.

We plug the expansion ( 15) the differential equations for the parameters  $\lambda, \gamma$  and  $z$  to obtain expansions for  $\dot{\lambda}, \dot{\gamma}$  and  $\dot{z}$  in terms of  $z$  and  $\bar{z}$ .

In the second step we transform  $z$  to a parameter  $y$  which satisfies a simpler different equation. We construct a polynomial  $P(z, \bar{z})$  with real coefficients and the smallest degree  $\geq 2$ , such that the new parameter  $y := z + P(z, \bar{z})$  satisfies the equation

$$\dot{y} = i\epsilon(\lambda)y + \sum_{2 \leq m+n \leq 2N+1} Y_{m,n}(\lambda)y^m \bar{y}^n + \text{Remainder}. \quad (17)$$

where the coefficients  $Y_{m,n}(\lambda)$  are purely imaginary, and  $Y_{m,n}(\lambda) = 0$  if  $m \neq n+1$ , and the term *Remainder* admits the estimate

$$|\text{Remainder}| \leq ct^{-\frac{2N+1}{2N}} \text{ or } |y(t)|^{2N+1} + |y(t)|^N \|\langle x \rangle^{-\mu} R_N\|_2 + \|\langle x \rangle^{-\mu} R_N\|_2^2.$$

Expressing the variables  $z$  and  $\bar{z}$  as power series in  $y$  and  $\bar{y}$  and plugging the result into the  $z$ -expansions for  $\dot{\gamma}$ ,  $\dot{\lambda}$  and  $\vec{R}$  mentioned above we obtain the  $y$ -expansions for these quantities.

As was mentioned above, the decrease of  $z$  (or  $y$ ) and therefore the relaxation of the soliton to its equilibrium occurs due to the radiation of the excess of the energy to infinity. The latter is possible only if the periodic solutions to the linearized equation are coupled to its continuous spectrum. To detect this coupling we must obtain an expansion of  $\vec{R}$  in the parameters  $y$  and  $\bar{y}$  up to the order  $2N$ . This is done in the third, most involved step. As in the first step we determine the coefficients  $R_{m,n}(\lambda)$  of the  $y$ -expansion of  $\vec{R}$  by solving the equations

$$[L(\lambda) - i\epsilon(\lambda)(m-n)]R_{m,n}(\lambda) = -P_c f_{m,n}(\lambda)$$

for certain functions  $f_{m,n}(\lambda)$  (see below). Recall that the number  $N$  is defined by the properties

$$\begin{aligned} i\epsilon(\lambda)(m-n) &\notin \sigma(L(\lambda)) \text{ if } |m-n| \leq N, \\ &\in \sigma(L(\lambda)) \text{ if } |m-n| > N. \end{aligned}$$

Thus for  $N < m+n \leq 2N$  the parameter  $i\epsilon(\lambda)(m-n)$  might be in the spectrum of  $L(\lambda)$ . To deal with this case we sort out the pairs  $(m, n)$  into "non-resonant pairs" satisfying  $|m-n| \leq N$  and "resonant pairs" satisfying  $|m-n| > N$ . For "non-resonant" pairs the operators

$$L(\lambda) - i\epsilon(\lambda)(m-n) : P_c \mathcal{L}^2 \rightarrow P_c \mathcal{L}^2$$

are invertible and for resonant pairs they are not (one has to change spaces in the latter case).

In the first two steps we expanded in  $z$  and  $\bar{z}$  (and in  $y$  and  $\bar{y}$ ) until  $m+n \leq N$  and consequently all the pairs,  $(m, n)$ , involved were non-resonant ones. Now our expansion involves pairs  $(m, n)$  with  $m+n > N$ , which include resonant pairs. A key point which we show is that for the subsets of pairs  $(m, n)$ ,  $m+n > N$ , determined by the inequality

$$m, n \leq N,$$

the terms  $f_{m,n}(\lambda)$  involve only "non-resonant" pairs (here we use that the parameter  $y$  satisfies (17) with  $Y_{m,n}(\lambda) = 0$  for  $m \neq n + 1$ ) and consequently we are able to solve for the coefficients  $R_{m,n}(\lambda)$  also in this case.

Finally, estimates of the remainders as in Equations (15) are done by rewriting the differential equations for them in an integral form (using the Duhamel principle) and using estimates of linear propagators derived in [BP1, BP2, BuSu, RSS, GoSc, GS1, GS2].

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